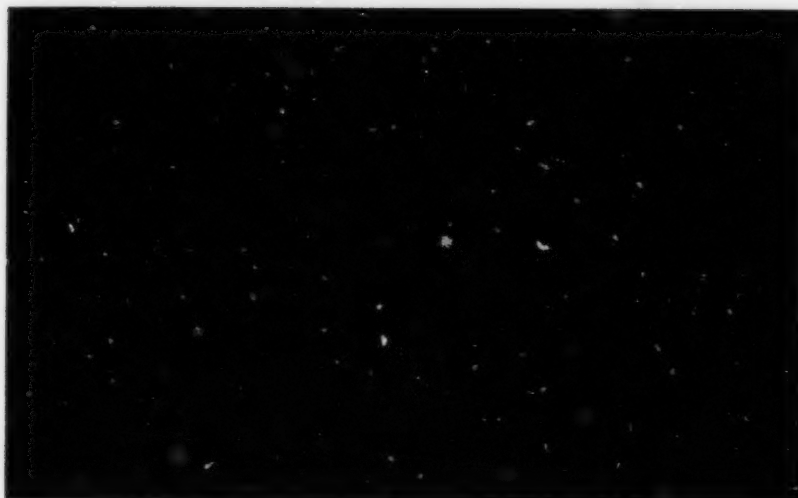




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**Transit Equilibrium Assignment:
A Model and Solution Algorithm**

by
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ABSTRACT

In this paper we propose a model for the transit equilibrium assignment problem (TEAP) and develop two algorithms for its solution. The behavior of transit users is modelled by using the concept of hyperpaths (strategies). The model is stated on a network which is obtained from the physical transit network by a transformation which makes explicit the transit generalized cost on walk, wait, in-vehicle and transfer/alight arcs. The waiting cost is a function of both the inter-arrival time of transit vehicles and congestion effect due to queues at stops. The TEAP is stated and formulated as a variational inequality in the space of hyperpaths and then solved iteratively by the linearized Jacobi and the projection methods. We prove that for strongly monotone cost mappings the linearized Jacobi method is globally convergent. The implementation of the algorithms and computational experiments are presented as well.

Keywords: transit assignment, traffic assignment, equilibrium problem, variational inequality, linearized Jacobi method, projection method.

RÉSUMÉ

Dans cet article nous proposons un modèle pour le problème d'affectation d'équilibre pour les voyageurs sur un réseau de transport en commun (PAETC) et nous développons deux algorithmes pour sa résolution. Le comportement de voyageurs est modélisé en utilisant le concept de hyperchemins (stratégies). Le modèle est formulé en utilisant une transformation du réseau physique qui explicite les liens de marche, d'attente, de voyage à bord d'un véhicule et de correspondance. La fonction de temps d'attente dépend de l'intervalle d'attente entre l'arrivée des véhicules et des effets de congestion dues aux files d'attente aux arrêts. Le PAETC est formulé en utilisant une approche d'inégalité variationnelle dans l'espace des hyperchemins et, ensuite, résolu par la méthode du Jacobi linéarisé et la méthode de projection. Nous démontrons que pour des applications fortement monotones, la méthode du Jacobi linéarisé converge globalement. L'implantation des algorithmes ainsi que des résultats de calcul expérimentaux sont également présentés.

Mots-clés: affectation de transport en commun, affectation du trafic, problème d'équilibre, approche d'inégalité variationnelle, méthode du Jacobi linéarisé, méthode de projection.

1 Introduction

We consider the formulation, analysis and algorithmic resolution of a transit equilibrium assignment problem (TEAP). The purpose of the TEAP is to model the flow of passengers on a congested transit network, induced by the travel demands which are given by a fixed origin-destination matrix. The transit network that corresponds to the TEAP consists of the underlying road highway network and the transit lines that use it.

The transit assignment problem has been studied recently by several authors. The main differences among various transit assignment methods are the travel hypotheses made, explicitly or implicitly, on the user's behaviour, with or without consideration of congestion effects. For uncongested transit networks, several algorithms have been developed. A good survey of these methods is given by Spiess and Florian (1989). To represent adequately the behavior of users waiting at bus stops shared by distinct lines (common lines), all models assume some sort of "strategies" that govern the choices made by the users. Given that the only information available to users is whether a bus has actually arrived or not, a strategy consists of a set of (deterministic) rules that dictates the (random) route of a passenger, from its origin to its destination. An uncongested transit assignment is formulated (Spiess and Florian, 1989), based on the concept of the optimal strategy, as a linear programming problem and solved by a polynomial time algorithm. Nguyen and Pallotino (1985) characterized and studied in detail the notion of the strategy, by using a graph theoretical approach, and developed a concept of a hyperpath. A strategy corresponds a hyperpath, i.e., a subnetwork containing all arcs of the corresponding strategy. At a bus stop, passengers always board the first vehicle to arrive from the set of attractive lines characterizing the hyperpath at that stop. The demand at a node is then assigned to the hyperpath in proportion to the fixed frequencies of lines in the attractive set of hyperpath.

Although this class of models can be useful and efficient to study transit systems where the impact of congestion effects on the route choice is low, they are unsuitable in the simulation of transit flows in some cities, such as Shanghai. High congestion levels may be so severe that they affect the passenger's route choices.

Several attempts have been carried to develop TEAP models that consider congestion effects. TRANSEPT (Last and Leak, 1976) uses an iterative process in order to assign trips to common lines, taking into account capacities and waiting times. Nevertheless, the trip loading process is per-

formed link by link for each route, which makes the model appropriate only for radial networks. Spiess and Florian (1989) give a general version of a nonlinear model in which in-vehicle travel times (or generalized travel costs) are increasing functions of passenger flow (called "discomfort functions"). The main limitation of these approaches is that waiting times at stops are not affected by congestion. A similar formulation was proposed by Nguyen and Pallotino (1985). They show that, by explicitly defining the flow and cost structures of the hyperpath, the standard equilibrium traffic assignment model can be adapted to transit networks by substituting the hyperpath flow space for the path flow space. As in Spiess and Florian's model, the waiting times are assumed constant. Gendreau (1984) and De Cea and Fernández (1988) extended the TEAP to include the consideration of congestion effects not only on the in-vehicle cost but also on the waiting time. The first formulates the waiting times with queueing theory models on an expanded network, while the second models the waiting time as a direct function of the flow-dependent effective frequency; only a solution algorithm for a simplified version of the model is presented.

Our approach is to adopt the concept of hyperpath in the TEAP formulation and apply it on a general network representation of the physical transit network. This general network, where each arc is assigned a frequency and a travel cost function, is a generalization of the traditional physical network as stated by Spiess and Florian (1989). A transformation matrix, i.e., an arc-hyperpath incidence matrix, from the hyperpath flow to the arc flow, is composed of elements with values between 0 and 1. Recall that the elements of an arc-path incidence matrix are restricted to be 0-1 values. The formulation of the waiting cost is also different from the existing models, in that the waiting cost is equal to the sum of a function of the flow independent frequencies and a function of the waiting and in-vehicle flows.

It is well known that the traffic equilibrium assignment and the TEAP can be formulated as variational inequalities. Two main approaches have been investigated for the solution of variational inequalities on transportation networks, i.e., the iterative approach, where the original problem is replaced by a sequence of optimization problems (Pang and Chan, 1982 and Florian and Spiess, 1982) and the optimization approach, where the problem is formulated as a mathematical program (Nguyen and Dupuis, 1984, Marcotte, 1985 and Marcotte and Guélat, 1988). For a more recent survey see Harker and Pang (1990).

Our solution method belongs to the first category and resorts to the successive resolution of the linearized optimization problems. The methods

are distinguished from one another according to the particular form of linearization used. The linearized Jacobi and projection methods are special instances of it. In this paper we will borrow some of the ideas presented in Dafermos (1980, 1983), Bertsekas and Gafni (1982) and Pang and Chan (1982). We will in particular show that strong monotonicity in the arc space (not in the hyperpath space !) is sufficient to obtain global convergence and uniqueness (in arc flow space) results for our model.

The main contribution of the paper is twofold:

1. A theoretical and computational investigation of the TEAP, including the new definition of waiting cost, using the linearized Jacobi and the projection methods in the hyperpath flow space;
2. Under the usual assumptions, a proof of global convergence of the linearized Jacobi and projection methods that does not require strong monotonicity in the hyperpath flow space.

The rest of the paper is organized as follows. Section 2 presents the basic model notation and the formulation. In section 3, we give a general linearized method with special attention to both the Jacobi and the projection methods. Section 4 is devoted to the study of the impact of model parameters on algorithmic convergence and uniqueness of the solution. Implementation and computational results are reported in Section 5. Finally, some conclusions are presented in Section 6.

2 Notation and Formulation

2.1 Basic notation and assumptions

Consider a transit network consisting of a set of distinct transit lines, transit stops and walk arcs, transfer and alight arcs and centroids where demand is generated and attracted. The network can be transformed into a general network where a typical stop node served by several transit lines is represented by a subgraph, in which each distinct wait arc is connected to a transit line. Such a subgraph is illustrated in Fig. 2.1.1. This general network is defined as $G = \{N, A\}$ where $N = \{n^i | i = 1, 2, \dots, |N|\}$ is the node set and $A = \{(t(a), h(a)) | a = 1, 2, \dots, |A|, t, h \in N\}$ is the arc set where $t(a)$ and $h(a)$ are the tail and head respectively of the arc a . Let $s(v) = \{s_a(v) | a = 1, 2, \dots, |A|\}$ be a vector of arc cost functions where v is the arc flow vector and let $f = \{f_a | a = 1, 2, \dots, |A|\}$ be the set of arc frequencies. The general network has four kinds of arcs: walk arc, wait arcs, on-board arcs and transfer/alight arcs represented by respective arc sets

A^1, A^2, A^3 and A^4 . Frequencies for the walk, in-vehicle and transfer/alight arcs can be infinite. Furthermore each travel arc is associated with exactly one transit line (one way) and there are no two nodes connecting more than two arcs and every wait arc is associated exactly with a transit line.

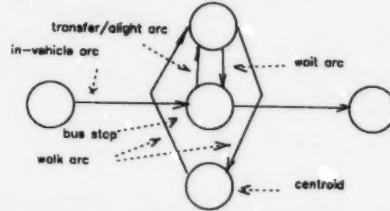


Fig.2.1.1. Representation of a transit stop in the general network

Now let us recall briefly the hyperpath concept. The hyperpath generalizes the concept of path; at each node of the network, a discrete variable specifies which outgoing arc will be travelled. A path may be viewed as a hyperpath for which the variable associated with a node can assume but one value, namely the index number of the successor of the node in the path. Thus a given hyperpath k connecting a pair of nodes $(p, q) = i$ is a subgraph $G^k = \{N^k, A^k\}$. Let P_a^k denote the (conditional) probability that arc a is travelled, given that the flow on the hyperpath k has reached node $t(a)$. In the context of transit assignment we have

$$P_a^k = \frac{f_a}{\sum_{a' \in FS_k(t(a))} f_{a'}} \quad (1)$$

where $FS_k(t(a))$ is the forward star set of arcs at the node $t(a)$ in the hyperpath k . Let us define $\alpha_{t(a)}^k$ to be the proportion of one unit flow travelling hyperpath k at node $t(a)$; δ_a^k to be the proportion of one unit flow travelling hyperpath k on arc a . We then have the basic relationships for a hyperpath from origin node p to destination node q :

$$\alpha_p^k = 1$$

$$\alpha_q^k = 1$$

$$\delta_a^k = \alpha_{t(a)}^k P_a^k$$

$$\sum_{a' \in BS_k(h(a))} \alpha_{t(a)}^k P_a^k = \alpha_{h(a)}^k$$

where $BS_k(h(a))$ is the backward star set of arcs at the node $h(a)$ in the hyperpath k .

When a passenger is travelling over a given hyperpath at the node t (bus stop), he considers a set of attractive arcs corresponding to a set of transit lines (called attractive lines) and boards the first vehicle to arrive among the lines represented by those arcs. The waiting cost at the node includes a combination of two factors: the time that the passenger waits until one of the buses of those attractive arcs arrives, i.e., $\epsilon / \sum_{a \in FS_k(t)} f_a$, and the time that the passenger waits before boarding on the bus, i.e., $s_a(v)$ on the arc where ϵ is a parameter whose value depends on the distribution assumed for the buses interarrival time, i.e., 1 in case of exponential distribution or 0.5 in case of uniform distribution. The cost of a hyperpath k is defined as follows:

$$\begin{aligned} S_k &= \sum_{a \in A} \delta_a^k [s_a(v) + w_{ak}] = \sum_{a \in A} \delta_a^k [s_a(v) + w_a P_a^k] \\ &= S_k^1 + S_k^2 + S_k^3 + S_k^4 \end{aligned} \quad (2)$$

where $w_{ak} = \epsilon / \sum_{b \in FS_k(t)} f_b$, $w_a = \epsilon / f_a$ and $P_a^k = f_a / \sum_{b \in FS_k(t)} f_b$

$S_k^1 = \sum_{a \in A^1} \delta_{ak} s_a(v)$ is the walk cost on k

$S_k^2 = \sum_{a \in A^2} \delta_{ak} s_a(v) + \sum_{a \in A} \delta_{ak} w_{ak}$ is the wait cost on k

$S_k^3 = \sum_{a \in A^3} \delta_{ak} s_a(v)$ is the on-board cost on k

$S_k^4 = \sum_{a \in A^4} \delta_{ak} s_a(v)$ is the transfer/alight cost on k

$$A = A^1 \cup A^2 \cup A^3 \cup A^4$$

or in a vector notation, we obtain:

$$S = \delta^t [s(v) + w] \quad (3)$$

where δ is the arc-hyperpath matrix with elements δ_a^k , $s(v)$ is a vector of arc cost functions and w is a vector of w_a 's.

2.2 The structure of the cost functions

We consider a possible structure of the cost functions for the four kinds of arcs. As stated in Section 2.1, two attributes are associated with every arc, that is, $s_a(v)$ and $w_a = \epsilon / f_a$.

Walk arcs

We assume that the costs on the walk arcs are independent of flow and given by $s_a = \alpha_1 t_a$, where α_1 is a positive parameter and t_a is a constant walk time; f_a is set to be ∞ .

Wait arcs

The costs on the wait arcs are dependent on both the flows that correspond to waiting for the bus (v_a) and the direct flows (v_d) corresponding to through passengers that share the bus capacity with the waiting flow. The functional form chosen is as follows:

$$s_a(v_a, v_d) = \alpha_2 [(v_a + \beta_2 v_d)/K]^\rho \quad (4)$$

where α_2 , β_2 and ρ are positive parameters and K is the bus capacity. According to the definition, we have $v_b = v_a + v_d$, where v_b is the in-vehicle flow.

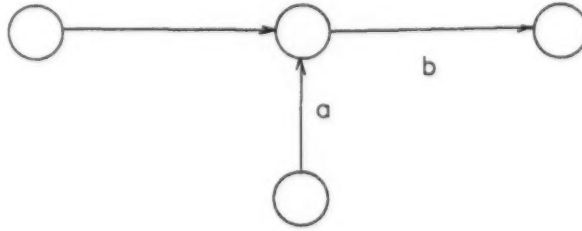


Fig.2.2.1

and (4) may be rewritten as

$$s_a(v_a, v_b) = \alpha_2 [(1 - \beta_2)v_a + \beta_2 v_b]/K]^\rho \quad (5)$$

with $0 \leq \beta_2 \leq 1$. In this case, f_a represents the bus frequency on arc b .

In-vehicle arcs

The costs of the in-vehicle arcs, t_b , and the discomfort costs depend on the direct flow, v_d , and wait flow, v_a , which is the total flow waiting for the bus. Thus we have the following relation as illustrated in Fig.2.2.1:

$$s_b(v_d, v_a) = \alpha_3 t_b + \beta_3 [(v_d + \gamma_3 v_a)/K]^\rho \quad (6)$$

where $\alpha_3, \beta_3, \gamma_3$ and ρ are the positive parameters associated with the in-vehicle arc. K is the capacity of the bus on arc a . Since $v_d = v_b - v_a$, we have

$$s_b(v_a, v_b) = \alpha_3 t_b + \beta_3 [(v_b + (\gamma_3 - 1)v_a)/K]^p. \quad (7)$$

We set f_b to be ∞ .

Transfer and alight arcs

We assume that the arc costs are independent of arc flows and given by a constant walk time t_a , i.e., $s_a = \alpha_4 t_a$ and let the frequencies f_a be infinite, i.e., $f_a = \infty$.

2.3 Travel behaviour

The set of feasible hyperpath flows Ω is defined as follows:

$$\sum_{k \in K_i} h_k = g_i, \quad i \in I, \quad (8)$$

$$h_k \geq 0, \quad k \in K_i, i \in I, \quad (9)$$

where h_k is the flow along hyperpath k , g_i is the travel demand from origin p to destination q , $(p, q) = i$, K_i is the set of hyperpaths from origin p to destination q , and I is the set of origin-destination pairs.

We assume that the passenger flows on the general network satisfy Wardrop's user-optimal equilibrium conditions (Wardrop, 1952), i.e.,

$$S_{k^*} \begin{cases} = u_i^* & \text{if } h_{k^*} > 0 \\ \geq u_i^* & \text{if } h_{k^*} = 0 \end{cases} \quad (10)$$

where u_i^* is the minimum hyperpath cost from p to q and S_{k^*} the equilibrium hyperpath cost on k^* . It can be shown that the equilibrium conditions (10) can be alternatively stated as the following variational inequality:

$$VI(\Omega, S): \quad S(h^*)^t(h - h^*) \geq 0 \quad \text{for all } h \in \Omega \quad (11)$$

or

$$s(v^*)^t(v - v^*) + W^t(h - h^*) \geq 0 \quad \text{for all } v \in \Omega_v \text{ and for all } h \in \Omega \quad (12)$$

where h^* and v^* are the equilibrium hyperpath and arc flows respectively; Ω_v is a corresponding feasible arc flow set.

$$v_a = \sum_i \sum_k \delta_{ak} h_k \quad (v = \delta h) \quad (13)$$

$$W_k = \sum_{a \in A} \delta_{ak} w_{ak} \quad (W = \delta w) \quad (14)$$

Theorem 2.3.1 Assume that $s(v)$ is continuous and Ω is a compact and convex set. There is at least one solution h^* of $VI(\Omega, S)$.

Proof. Due to the assumption on $s(v)$, we know that

$$S_k(h) = \sum_{a \in A} \delta_{ak} [s_a(\delta h) + w_{ak}]$$

is a positive continuous function of h . This ensures the existence of a solution h^* according to Smith (1979,1983). $\square$

Theorem 2.3.2 Consider the variational inequality $VI(\Omega, S)$. v^* and $W^t h^*$ are unique if $s(v)$ is a strictly monotone function, i.e.,

$$[s(v') - s(v'')][v' - v''] > 0 \text{ for all } v', v'' \in \Omega_v \text{ and } v' \neq v''. \quad (15)$$

Proof. Let us suppose that we have two solutions (v', h') and (v'', h'') . Then we obtain

$$s(v')^t(v'' - v') + W^t(h'' - h') \geq 0$$

$$s(v'')^t(v' - v'') + W^t(h' - h'') \geq 0.$$

Adding them up, we have

$$[s(v') - s(v'')][v' - v''] \leq 0.$$

Due to (15), we have $v' = v''$. Since h' and h'' satisfies the user-optimal equilibrium condition, we have $u'^* = u''^*$. Then

$$\begin{aligned} \sum_i \sum_k S_k(h') h'_k &= \sum_i \sum_k S_k(h'') h''_k \\ \sum_a s_a(v') v'_a + \sum_i \sum_k W_k h'_k &= \sum_a s_a(v'') v''_a + \sum_i \sum_k W_k h''_k \Rightarrow \\ \sum_i \sum_k W_k h'_k &= \sum_i \sum_k W_k h''_k \end{aligned}$$

in compact form, we have

$$Wh' = Wh''. \quad \square$$

This result is given without proof in (Nguyen and Pallotino, 1985).

3 A General Linearized Method

3.1 The linearized method

A general linearized method for solving $VI(\Omega, S)$ may be stated as follows: Given a hyperpath flow $h^l \in \Omega$, h^{l+1} is defined as a solution to a variational inequality subproblem $VI(\Omega, S^l)$ where

$$S^l(h) = S(h^l) + \frac{B(h^l)}{\alpha}(h - h^l) \quad (16)$$

where $B(h^l)$ is a square matrix and α is a positive parameter.

This class of methods include the linearized Jacobi method, where $B(h^l)$ is the diagonal of the Jacobian matrix of $S(h^l)$, i.e., $\nabla S(h^l)$, and the projection method, where $B(h^l) = B$ is some fixed symmetric positive definite matrix.

In both cases, each subproblem is well defined and has been proved to be equivalent to solving a strict convex quadratic minimization problem, due to the properties of $B(h^l)$, i.e.,

$$\min \int_0^h S^l(x) dx \quad (17)$$

$$\text{s.t. } h \in \Omega$$

or

$$\min (h - h^l)^t s(h^l) + (h - h^l)^t \frac{B(h^l)}{2\alpha} (h - h^l) \quad (18)$$

$$\text{s.t. } h \in \Omega.$$

For completeness we recall the following theorem.

Theorem 3.1.1 Define the mapping P as $h^{l+1} = P(h^l)$. Then h^* is a solution to $VI(\Omega, S)$ if and only if it is a fixed point of P .

3.2 The linearized Jacobi method (LJM)

3.2.1. Theoretical analysis

By definition, $B(h)$ is the diagonal of $\nabla S(h)$.

Proposition 3.2.1
$$\frac{\partial S_k}{\partial h_k} = \sum_{a \in A} \delta_{ak}^2 \frac{\partial s_a(v)}{\partial v_a}.$$

Proof. Since $S_k = \sum_{a \in A} \delta_{ak}(s_a(v) + w_{ak})$ and $v_a = \sum_i \sum_k \delta_{ak} h_k$, we have

$$\frac{\partial S_k}{\partial h_k} = \frac{\partial S_k}{\partial v_a} \frac{\partial v_a}{\partial h_k} = \sum_{a \in A} \delta_{ak}^2 \frac{\partial s_a(v)}{\partial v_a}. \quad \square$$

Let u_p be the cost of hyperpath k from p to q , u_r be the cost of hyperpath $\bar{k}$ from r to q where $\bar{k}$ is a subhyperpath of k and f_p be the cumulative frequency of the hyperpath k at node p , i.e., $f_p = \sum_{b \in FS_k(p)} f_b$.

Then we have the following recursive formula ($a = (p, r)$): for every $p \in N$,

$$S_k = u_p = \sum_{a \in FS_k(p)} \left(\frac{f_a}{f_p} \right) (u_r + s_a(v) + w_{ak}),$$

$$u_q = 0,$$

and obviously

$$\frac{\partial S_k}{\partial h_k} = \frac{\partial u_p}{\partial h_k} = \sum_{a \in FS_k(p)} \left(\frac{f_a}{f_p} \right)^2 \left(\frac{\partial u_r}{\partial h_{\bar{k}}} + \frac{\partial s_a(v)}{\partial v_a} \right),$$

$$\frac{\partial u_q}{\partial h_k} = 0.$$

In order to present the following convergence theorem, we shall use the definitions, notations and propositions given in Appendix A.

Theorem 3.2.1 (Convergence Theorem) *Let Ω be a nonempty compact and convex set. Let $s(v)$ be continuous. Let $B(h)$ be a symmetric positive definite matrix. Then there exists a symmetric positive definite matrix $\bar{B}$ with $B(h) - \bar{B}$ positive semi-definite for all $h \in \Omega$. If there is a parameter $0 < \theta < 1$ such that*

$$\|\bar{B}^{-1} \{ \alpha [S(\bar{h}(h)) - S(h)] - B(h)[\bar{h}(h) - h] \}\|_{\bar{B}} \leq \theta \|\bar{h}(h) - h\|_{\bar{B}} \quad (19)$$

holds for all $h \in \Omega$ where

$$\bar{h}(h) = \arg \min \{ \|h - y\|_{\bar{B}} | y \in H^* \cap \Omega, \} \quad (20)$$

and $H^* = \{h | v^* = \delta h\}$ and v^* is a unique solution, then the sequence of iterates $\{h^l\}$ generated by the linearized Jacobi method converges to a solution $\bar{h}$ of the variational inequality $VI(\Omega, S)$ for any initial vector $h^1 \in \Omega$ with the estimate

$$\|h^l - \bar{h}\|_{\bar{B}} \leq \frac{(\theta + 1)\theta^{l-1}}{(1 - \theta)} \|h^1 - \bar{h}(h^1)\|_{\bar{B}}. \quad (21)$$

Proof.

The existence of $\bar{B}$ is obvious since $\bar{B}$ can be selected such that

$$\bar{B} = \lambda_{\min} I$$

where $\lambda_{\min}$ is the minimum eigenvalue of $B(h)$ and I is a corresponding identity matrix.

From Theorem 2.3.1, we know that in this case at least one solution $\bar{h}$ exists. Note that the matrix $B(h)$ is a diagonal positive symmetric definite for all $h \in \Omega$. Thus each $P(h)$ is well defined. Since $P(h)$, $\bar{h}(h)$ solve $VI(\Omega, \bar{S}(h))$ where $\bar{S}(h) = S(h) + \frac{B(h)}{\alpha}(P(h) - h)$, and $VI(\Omega, S)$ respectively, we have

$$[S(h) + \frac{B(h)}{\alpha}(P(h) - h)]^t [\bar{h}(h) - P(h)] \geq 0$$

$$S(\bar{h}(h))^t [P(h) - \bar{h}(h)] \geq 0$$

Adding the two inequalities, we get

$$[P(h) - \bar{h}(h)]^t \cdot B(h)[P(h) - \bar{h}(h)] \leq$$

$$[P(h) - \bar{h}(h)]^t \{\alpha[S(\bar{h}(h)) - S(h)] - B(h)[\bar{h}(h) - h]\}.$$

Therefore:

$$\|P(h) - \bar{h}(h)\|_{\bar{B}}^2$$

$$\leq \|P(h) - \bar{h}(h)\|_{\bar{B}} \cdot \|B^{-1}\{\alpha[S(\bar{h}(h)) - S(h)] - B(h)[\bar{h}(h) - h]\}\|_{\bar{B}}$$

$$\leq \theta \|P(h) - \bar{h}(h)\|_{\bar{B}} \|\bar{h}(h) - h\|_{\bar{B}}.$$

Dividing by $\|P(h) - \bar{h}(h)\|_{\bar{B}}$, there comes

$$\|P(h) - \bar{h}(h)\|_{\bar{B}} \leq \theta \|h - \bar{h}(h)\|_{\bar{B}}. \quad (22)$$

We call (22) the contraction property. Let $k > l \geq 1$. From the contraction property and the triangle inequality, we obtain

$$\begin{aligned}
& \|h^k - h^l\|_{\overline{B}} \\
& \leq \sum_{j=l}^{k-1} \|h^{j+1} - h^j\|_{\overline{B}} \\
& \leq \sum_{j=l}^{k-1} \|h^{j+1} - \overline{h}(h^j)\|_{\overline{B}} + \|h^j - \overline{h}(h^j)\|_{\overline{B}} \\
& \leq \sum_{j=l}^{k-1} (\theta + 1) \|h^j - \overline{h}(h^j)\|_{\overline{B}} \\
& \leq \sum_{j=l}^{k-1} (\theta + 1) \|h^j - \overline{h}(h^{j-1})\|_{\overline{B}} \\
& \leq \sum_{j=l}^{k-1} (\theta + 1) \theta \|h^{j-1} - \overline{h}(h^{j-1})\|_{\overline{B}} \\
& \leq \sum_{j=l}^{k-1} (\theta + 1) \theta^{j-1} \|h^1 - \overline{h}(h^1)\|_{\overline{B}} \\
& \leq (\theta + 1) \frac{\theta^l - \theta^k}{\theta[1 - \theta]} \|h^1 - \overline{h}(h^1)\|_{\overline{B}}.
\end{aligned}$$

Since $0 < \theta < 1$, it follows that $\{h^l\}$ is a Cauchy sequence and must therefore converge to, say, $\overline{h}$ due to the compactness assumption. Being a contraction,

$$P(\overline{h}) = \lim_{l \rightarrow \infty} P(P(h^l)) = \lim_{l \rightarrow \infty} P(h^{l+1}) = \overline{h}.$$

This shows that $\overline{h}$ is indeed a fixed point of P . By Theorem 3.1.1, $\overline{h}$ is a solution of $VI(\Omega, S)$. Finally, letting $k \rightarrow \infty$, we arrive at the estimate, since $\lim_{k \rightarrow \infty} \theta^k = 0$. $\square$

We are now ready to look into the requirement (19) for convergence and the physical meaning of θ, α .

Consider the range $R(\delta)$ of δ , the null space $U(\delta)$ of δ , and its orthogonal complement

$$U(\delta)^\perp = \{z \in \Omega | zx = 0, \forall x \in U(\delta)\}$$

For all $h \in \Omega$, $v = \delta h$ and

$$\bar{h}(h) = \arg \min \{\|h - y\|_{\bar{B}} | y \in H^*\}$$

we have

$$v - v^* = \delta(h - \bar{h}(h)) \text{ and } h - \bar{h}(h) \in U(\delta)^\perp, \quad \forall v - v^* \in R(\delta).$$

When viewed as a mapping from $U(\delta)^\perp$ to $R(\delta)$, δ is one-to-one and onto, i.e., δ is invertible.

Proposition 3.2.2 (Bertsekas and Gafni, 1982) Assume

$$\bar{h}(h) = \arg \min \{\|h - y\|_{\bar{B}} \mid y \in H^*\}.$$

Then there exists a positive scalar η such that for all $h \in \Omega$,

$$\|h - \bar{h}(h)\|_{\bar{B}} \geq \eta \|h - \bar{h}(h)\|_{\bar{B}} \quad (23)$$

Proposition 3.2.3 The requirement (19) is satisfied if

$$\|s(v') - s(v'')\|_2 \leq L_s \|v' - v''\|_2, \quad (24)$$

$$[s(v') - s(v'')]^\top (v' - v'') \geq \lambda \|v' - v''\|_2^2 \quad (25)$$

and $B(h)$ is a symmetric positive definite matrix for all $h \in \Omega$ with $1 > \theta^2 \geq \omega - 2\alpha\lambda\gamma\eta + \alpha^2 L_s^2 \beta^2 > 0$

where

L_s is a positive parameter

α is a positive parameter

λ is the minimum eigenvalue of the symmetric part of $\nabla s(v)$ over a feasible region which is larger than 0

β is defined in Proposition A.3 to be $\|\delta\|_F^2 \Lambda_{\min}^{-1}$

$\Lambda_{\max}$ is the maximum eigenvalue of $\bar{B}$

$\Lambda_{\min}$ is the minimum eigenvalue of $\bar{B}$

γ is defined in Proposition A.3 to be $\|\delta\|_F^2 \Lambda_{\max}^{-1}$

ω is a positive parameter larger than 1.

Proof. Let

$$A0 = \|\bar{B}^{-1} \{\alpha[S(\bar{h}(h)) - S(h)] - B(h)(\bar{h}(h) - h)\}\|_{\bar{B}}^2$$

$$\begin{aligned}
&= \alpha^2 [S(\bar{h}(h)) - S(h)]^t \cdot \bar{B}^{-1} [S(\bar{h}(h)) - S(h)] \\
&\quad - 2\alpha [S(\bar{h}(h)) - S(h)]^t \bar{B}^{-1} B(h) (\bar{h}(h) - h) \\
&\quad + (\bar{h}(h) - h)^t \cdot B(h) \bar{B}^{-1} B(h) (\bar{h}(h) - h) \\
&= A1 - A2 + A3
\end{aligned}$$

We have the following inequalities:

$$\begin{aligned}
A1 &\leq \alpha^2 \Lambda_{\min}^{-1} [S(\bar{h}(h)) - S(h)]^t [S(\bar{h}(h)) - S(h)] \quad \text{by (38)} \\
&\leq \alpha^2 \Lambda_{\min}^{-1} \|\delta\|_F^2 \|s(v^*) - s(v)\|_2^2 \quad \text{by (40)} \\
&\leq \alpha^2 \Lambda_{\min}^{-1} \|\delta\|_F^2 L_s^2 \|v^* - v\|_2^2 \quad \text{by (24)} \\
&\leq \alpha^2 \Lambda_{\min}^{-1} \|\delta\|_F^2 L_s^2 \|\delta\|_F^2 \Lambda_{\min}^{-1} \|\bar{h}(h) - h\|_B^2 \quad \text{by (44)} \\
&= \alpha^2 L_s^2 \beta^2 \|\bar{h}(h) - h\|_B^2 \quad \text{by definition of } \beta \\
&\leq \alpha^2 L_s^2 \beta^2 \|\bar{h}(h) - h\|_B^2 \quad \text{by definition of } \bar{h} \text{ and } \bar{h}.
\end{aligned}$$

$$\begin{aligned}
A2 &\geq 2\alpha [S(\bar{h}(h)) - S(h)]^t (\bar{h}(h) - h) \\
&= 2\alpha [s(v^*) - s(v)]^t (v^* - v) \quad \text{by Proposition 3.2.2} \\
&\geq 2\alpha \lambda \|v^* - v\|_2^2 \quad \text{by (25)} \\
&\geq 2\alpha \lambda \gamma \|\bar{h}(h) - h\|_B^2 \quad \text{by (45)} \\
&\geq 2\alpha \lambda \gamma \eta \|\bar{h} - h\|_B^2. \quad \text{by (23)}
\end{aligned}$$

$$\begin{aligned}
A3 &\geq (\bar{h}(h) - h)^t \cdot \bar{B}(\bar{h}(h) - h) \\
&= \|\bar{h}(h) - h\|_B^2.
\end{aligned}$$

There exists a positive parameter ω larger than 1 such that

$$A3 \leq \omega \|\bar{h}(h) - h\|_B^2.$$

Thus

$$A0 \leq \bar{\theta}^2 \|\bar{h}(h) - h\|_B^2 \leq \theta^2 \|\bar{h} - h\|_B^2$$

where

$$1 > \theta^2 \geq \bar{\theta}^2 = \omega - 2\alpha\lambda\gamma\eta + \alpha^2 L_s^2 \beta^2 > 0. \quad \square$$

In practice, it is difficult to determine α in advance since λ and Λ are both the eigenvalues of two very large matrices in real applications; $\bar{\alpha}$ can only be estimated by a trial and error.

3.2.2. Algorithmic development

In view of the nature of the block diagonal form of $B(h^l)$, the quadratic program (18), can be decomposed into a collection of smaller, independent quadratic programs, one per $i = (p, q)$ pair, i.e.,

$$\min \sum_{k \in K_i} [(h_k - h_k^l) S_k(h^l) + \frac{B_k(h^l)}{2\alpha} (h_k - h_k^l)^2] \quad (26)$$

$$\text{s.t. } \sum_{k \in K_i} h_k = g_i \quad (27)$$

$$h_k \geq 0 \quad \forall k \in K_i. \quad (28)$$

These programs can be solved easily (Kennington, 1980), if each set K_i contains only a few hyperpaths. In our case, however, the number of hyperpaths in K_i is very large. Thus it is preferable to start with a subset of each K_i and augment this subset as necessary (Bertsekas and Gafni, 1982). The resulting algorithm LJM is as follows:

Algorithm LJM

Step 1 (initialization): Let ϵ be a small positive parameter. Find $K_i^0 \in K_i, i \in I, h^0$ and v^0 ; Let $l = 1$.

Step 2 (shortest hyperpath): For every $i \in I$, find a shortest hyperpath k_i , using $s(v^{l-1})$ and f .

Output S_{k_i} and $\partial S_{k_i} / \partial h_{k_i}$ for all $k_i \in K_i^{l-1}, i \in I$.

Step 3 (hyperpath union): For every $i \in I$, do

$$K_i^l = K_i^{l-1} \cup k_i$$

Step 4 (cost evaluation): Calculate cost and first derivative of the cost for all hyperpaths other than k_i . We have S_{k_i} and $\partial S_{k_i}/\partial h_{k_i}$ for $k_i \in K_i^{l-1}, i \in I$.

Step 5 (solve quadratic program): For every $i \in I$, solve (26), (27), (28) and find h^l .

Step 6 (convergence test): If $\|h^l - h^{l-1}\| \leq \epsilon$, then stop.

Step 7 (update): $v^l = \delta h^l$ and $s(v^l) \leftarrow l + 1$ and return to step 2.

We recall that the shortest hyperpath problem has been studied in great depth by Spiess and Florian (1989) and Nguyen and Pallottino (1985, 1986) and can be solved efficiently by polynomial time algorithms.

At Step 5, one can use an alternative convergence measure at iteration l , CM^l , is:

$$CM^l = \sum_i \Delta h_i^l \Delta s_i^l / (g_i s_{min}^{i,l}) \quad (29)$$

where Δh_i^l is the portion of the demand that does not lie on the shortest paths of the i th pair at the end of iteration l ; g_i is the demand for the i th pair; Δs_i^l is the difference between the largest and smallest hyperpath cost for the i th pair at the end of iteration l ; $s_{min}^{i,l}$ is the shortest hyperpath cost of the i th pair at iteration l .

It is clear that the CM^l is zero if and only if the solution is an equilibrium flow.

3.3 A projection method (PM)

If $B(h) = B$ is some fixed symmetric positive definite matrix, then the algorithm LJM is the realization of a projection method (PM). Since the PM is a specialization of the LJM, we can easily conclude that Theorem 3.2.1, Propositions 3.2.2, and 3.2.3 and the corresponding algorithm still apply for the PM with $B(h) = \bar{B} = B$.

From the computational point of view, we expect the LJM to perform better than the PM, because at each iteration the function $S(h^l)$ is better approximated. This has been confirmed in the computational experiments presented in Section 5.

4 Structural Analysis of The Cost Function

The cost function parameters influence the model behaviour in mainly two respects:

1) The model output results. In real applications, the parameters can be adjusted so that the results from the model can be best fitted to the reality. This adjustment process is usually called model calibration.

2) Properties of the model. In our case, if some care is not taken, some assumptions would not be satisfied, especially the strong monotonicity assumption.

It is our hope that the range of parameters for which the calibrated functions satisfy the uniqueness and convergence conditions imposed on the model is large. This may not be the case in general.

Before discussing some properties of the model, we recall some basic results (see Ortega and Rheinboldt, 1970, for the proofs).

Proposition 4.0.1 *The arc cost function $s(v)$ is (strongly) monotone if its Jacobian matrix $\nabla s(v)$ is (positive definite) positive semidefinite over a compact and feasible region.*

Proposition 4.0.2 *A square matrix A is positive semidefinite (positive definite) if and only if the following two equivalent statements are true:*

- 1) *for any nonzero vector x , $x \cdot Ax \geq 0 (> 0)$;*
- 2) *all eigenvalues of the matrix $A + A^t$ are nonnegative (positive).*

If $-A$ is positive semidefinite (positive definite), then A is negative semidefinite (negative definite), then it is said to be indefinite.

A square matrix A is not definite otherwise.

Proposition 4.0.3 *Let a symmetric matrix A have the block diagonal structure:*

$$\begin{pmatrix} A_{11} & 0 & 0 & 0 \\ 0 & A_{22} & 0 & 0 \\ 0 & 0 & \cdot & 0 \\ 0 & 0 & \cdot & 0 \\ 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & A_{nn} \end{pmatrix}$$

where A_{ii} , $i=1,2,\dots,n$, are square symmetric matrices. A is positive (negative) definite if A_{ii} are positive (negative) definite for all $i=1,\dots,n$.

Proof. Straightforward. $\square$

Let us now inspect in detail the cost structure of the general network. We have that

- 1) the cost of every wait arc depends on the flow not only on itself but also on its associated travel arc;
- 2) the cost of every in-vehicle arc depends on its own flow and also on the flow on the wait arc associated with it if any;
- 3) all other arc costs only depend on their own flow.

Without loss of generality, we can write $\nabla s(v)$ as

$$J = \begin{pmatrix} J_{12} & 0 & 0 & 0 \\ 0 & J_{34} & 0 & 0 \\ 0 & 0 & \cdot & 0 \\ 0 & 0 & J_{m-1,m} & 0 \\ 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & J_{n-1,n} \end{pmatrix}$$

The matrices $J_{i,i+1}$, $i = 1, \dots, m-1$, are the 2-dimensional square submatrices with the indices 1, 3, 5, ..., $m-1$ corresponding to wait arcs and 2, 4, 6, ..., m to travel arcs. The matrix $J_{i,i+1}$, $i = m+1, \dots, n-1$, is in fact a 1-dimensional matrix, containing only one nonnegative element corresponding to the walk, in-vehicle and transfer/alight arcs. We will use a symmetric matrix $\bar{J} = (J + J^t)/2$ for the validation of the positive-definite matrix because of the relationship $x \cdot Jx = x \cdot 1/2(J + J^t)x = x\bar{J}x$. In view of Proposition 4.0.3, we only need to study $J_{i,i+1}$, $i = 1, \dots, m-1$. As a special case, we consider the cost structure given in Table 4.2.1. for the wait arc a and the in-vehicle arc b .

Wait arc (a)	In-vehicle arc (b)
$[(1 - \beta_2)v_a + \beta_2 v_b] / K$	$\alpha_3 t_b + [v_b + (\gamma_3 - 1)v_a] / K$
$\alpha_2 = 1, \rho = 1$	$\beta_3 = 1, \alpha_3 = 1, \rho = 1$

Table 4.2.1

The sub-Jacobian-matrix is given:

$$J_{ab} = \begin{pmatrix} (1 - \beta_2)/K & \beta_2/K \\ (\gamma_3 - 1)/K & 1/K \end{pmatrix} \quad (30)$$

and its symmetric part is as follows:

$$\bar{J}_{ab} = \begin{pmatrix} (1 - \beta_2)/K & (\beta_2 + \gamma_3 - 1)/2K \\ (\beta_2 + \gamma_3 - 1)/2K & 1/K \end{pmatrix}. \quad (31)$$

Let λ be the eigenvalues of $\bar{J}_{ab}$. We have

$$\lambda = [(2 - \beta_2) \pm \sqrt{(2 - \beta_2)^2 - 4(1 - \beta_2) + (\beta_2 + \gamma_3 - 1)^2}] / K \quad (32)$$

Proposition 4.0.4 $\bar{J}_{ab}$ is not negative definite.

Proof.

$$\lambda_{max} = [(2 - \beta_2) + \sqrt{(2 - \beta_2)^2 - 4(1 - \beta_2) + (\beta_2 + \gamma_3 - 1)^2}] / K > 0$$

where $0 \leq \beta_2 \leq 1$ by definition. $\square$

With a similar but more complicated approach, it is also possible to explore the monotonicity with respect to other parameters, in particular $\rho > 1$.

5 Implementation and Computational Experience

5.1 Implementation

A computer program written in Fortran 77 was developed on the SUN 3/50 (MC68020) workstations of the Center for Research on Transportation at University of Montreal to implement the LJM and the PM algorithms.

5.2 Experimental design

Consider the 4-zone physical transit network given in Fig. 5.2.1.

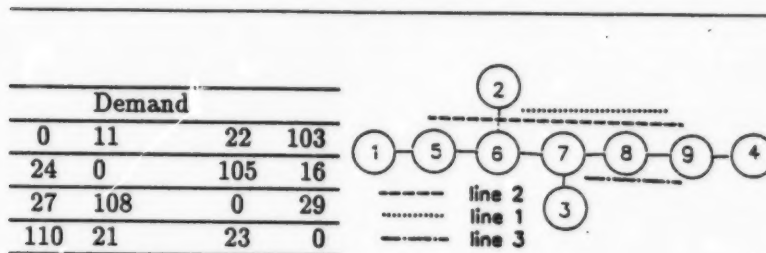


Fig. 5.2.1 A 4-zone physical transit network

The corresponding transformed network is given in Fig. 5.2.2. and the specific information of the general network is presented in Table 5.2.1. and the cost functions are defined as in Section 2. A brief description of three experiments is given in Table 5.2.2.

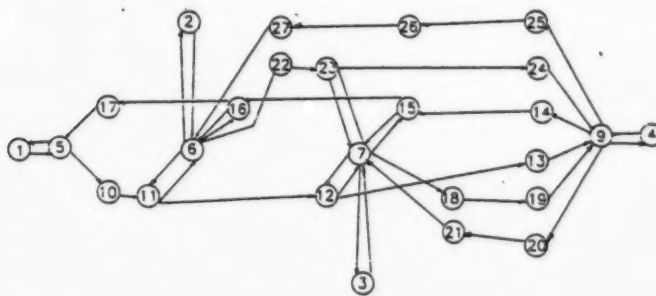


Fig. 5.2.2 4-zone general network

The results are analysed in terms of the number of iterations (IT), path volume differences (PVD) and the convergence measure (CM).

arc type	arc	t_a	$1/f_a$
walk	(1,5)	2.	∞
	(2,6)	2.	∞
	(3,7)	2.	∞
	(4,9)	1.	∞
	(5,1)	2.	∞
	(6,2)	2.	∞
	(7,3)	2.	∞
	(9,4)	1.	∞
wait	(5,10)	0.	5.00
	(6,11)	0.	5.00
	(6,16)	0.	5.00
	(6,22)	0.	3.00
	(7,12)	0.	5.00
	(7,15)	0.	5.00
	(7,18)	0.	4.00
	(7,23)	0.	3.00
	(7,26)	0.	3.00
	(9,24)	0.	5.00
	(9,20)	0.	4.00
	(9,25)	0.	3.00
travel	(10,11)	2.	∞
	(11,12)	2.	∞
	(12,13)	3.	∞
	(14,15)	3.	∞
	(15,16)	2.	∞
	(16,17)	2.	∞
	(18,19)	3.	∞

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	(20,21)	3.	∞
	(22,23)	2.	∞
	(23,24)	3.	∞
	(25,26)	3.	∞
	(26,27)	2.	∞
transfer/ align	(11,6)	3.	∞
	(12,7)	3.	∞
	(13,9)	3.	∞
	(15,7)	3.	∞
	(16,6)	3.	∞
	(17,5)	3.	∞
	(19,9)	3.	∞
	(21,7)	3.	∞
	(23,7)	3.	∞
	(24,9)	3.	∞
	(26,7)	3.	∞
	(27,6)	3.	∞

Table 5.2.1 Information on the general network

Experiment	Aims
1	Comparison between the PM and the LJM
2	Analysis of influence of the parameter α
3	Analysis of influence of the parameter ρ

Table 5.2.2 Description of three experiments

5.3 Computational Results

5.3.1. Experiment 1 (EX.1)

The parameters selected for both the PM and the LJM are listed in Table 5.3.1.1 and the corresponding computational results are listed in Table 5.3.1.2. In fact, EX.1 is based on an asymmetric case problem with an affine cost function. In this case, the LJM is much more efficient than the PM, as noted by Nagurney (1984). The condition $[s(v^l) - s(v^{l-1})][v^l - v^{l-1}] > 0$

was satisfied for every pair of two consecutive points in the convergence sequence, even though the corresponding Jacobian matrix may not be necessarily positive definite here. Another observation is that the criteria PVD and CM are not consistent until iteration 5.

method	α_1	α_2	β_2	α_3	β_3	γ_3	α_4	K	α	$B(h)$	ρ
PM	1.0	1.0	0.5	1.0	1.0	1.5	1.0	50	0.88	1	1
LJM	1.0	1.0	0.5	1.0	1.0	1.5	1.0	50	0.88	*	1

Table 5.3.1.1 The parameters for EX.1.

* $B(h)$ varies at each iteration.

IT.	PM		LJM	
	PVD	CM	PVD	CM
1	4.116	0.263×10^0	136.572	0.091×10^0
2	3.995	0.251×10^0	32.411	0.015×10^0
3	3.878	0.240×10^0	18.751	0.006×10^0
4	3.765	0.229×10^0	12.175	0.001×10^0
5	3.661	0.219×10^0	2.065	0.001×10^0
6	3.579	0.210×10^0	1.430	0.006×10^{-1}
7	3.499	0.202×10^0	1.016	0.004×10^{-1}
8	3.422	0.194×10^0	0.738	0.003×10^{-1}
9	3.342	0.186×10^0	0.546	0.002×10^{-1}
10	3.223	0.179×10^0	0.408	0.001×10^{-1}
11	3.144	0.172×10^0	0.308	0.001×10^{-1}
12	3.067	0.165×10^0	0.235	0.008×10^{-2}
...	...	...	...	...
19	2.600	0.125×10^0	0.038	0.001×10^{-2}
...	...	...	...	...
100	0.354	0.013×10^0		

Table 5.3.1.2. Computational results for EX.1.

5.3.2. Experiment 2 (EX.2)

The parameters are unchanged with the exception of α which is determined by trial and error for the LJM. The results of Table 5.3.2.1. show that the value 2.00 can be a good estimate of the optimal α in EX.2 and

suggests that a substantial decrease in the number of iterations can result from such a choice.

IT.	$\alpha = 1.5$		$\alpha = 2.00$		$\alpha = 3.00$	
	PVD	CM	PVD	CM	PVD	CM
1	181.60	0.063×10^0	212.800	0.051×10^0	261.594	0.033×10^0
2	50.378	0.004×10^0	58.658	0.002×10^0	113.178	0.008×10^0
3	6.190	0.009×10^{-1}	4.327	0.002×10^0	74.020	0.022×10^0
4	1.905	0.003×10^{-1}	0.668	0.008×10^{-1}	47.540	0.009×10^{-2}
5	0.891	0.001×10^{-1}	0.246	0.004×10^{-1}	24.434	0.013×10^0
6	0.489	0.001×10^{-1}	0.115	0.001×10^{-2}	14.200	0.004×10^{-1}
7	0.288	0.006×10^{-2}	0.056	0.008×10^{-2}	8.274	0.004×10^0
8	0.174	0.003×10^{-2}	0.027	0.003×10^{-2}	4.818	0.001×10^{-1}
9	0.107	0.002×10^{-2}	0.014	0.002×10^{-2}	2.805	0.001×10^0
10	0.066	0.001×10^{-2}	0.007	0.001×10^{-2}	1.633	0.006×10^{-2}

Table 5.3.2.1 Computational results for EX.2.

5.3.3. Experiment 3 (EX.3)

In this experiment, α is set to 0.25 and ρ varies from 1 to 3. The results (Table 5.3.3.1) suggest that the computational effort increases with ρ . Indeed, S is better approximated for small ρ values.

IT.	$\rho = 1.$		$\rho = 2.$		$\rho = 3.$	
	PVD	CM	PVD	CM	PVD	CM
1	43.882	0.209	156.029	1.720	212.616	5.295
2	36.745	0.124	60.872	0.664	59.491	1.580
3	29.224	0.071	39.860	0.367	19.899	0.749
4	14.186	0.047	21.379	0.215	12.988	0.449
5	12.501	0.031	11.946	0.177	8.720	0.801
6	11.083	0.019	9.341	0.151	8.689	0.708
7	5.745	0.013	9.199	0.137	9.630	0.656
8	5.043	0.011	9.075	0.125	9.324	0.583
9	4.665	0.008	8.963	0.116	8.943	0.520
10	4.319	0.007	8.860	0.108	8.555	0.467

Table 5.3.3.1 Computational results of EX.3.

6 Conclusions

In this paper, we considered the transit equilibrium assignment problem with a new way of modeling the flow dependent waiting time. For this problem, it seems that the LJM performs very well and converges after a few iterations, which confirms our theoretical study. The new model behaves very well, especially in a congested transit network, where the waiting time increases with the passenger volume.

In the current model, the distribution of flow on a hyperpath is flow independent, which may be a drawback of the model presented. However, if it is to be flow dependent, then we have a nonlinear relationship, i.e., $v = \delta(h)h$ in place of the linear transformation. This class of models needs to be studied more in detail. Also, due to the large size of the physical transit network, we will in the future investigate data structures as well as easily parallelizable decomposition schemes aimed at reducing the necessary computation time to obtain a solution.

Appendix A

In order to present a convergence theorem, we give the following definitions, notations and propositions.

If B is a symmetric positive definite matrix, the B -norm of a vector x is given by

$$\|x\|_B = (x^t \cdot Bx)^{1/2} \quad (33)$$

If B is the identity matrix, then the B -norm becomes the standard Euclidean norm noted by $\|\cdot\|_2$. The induced B -norm of a matrix G is

$$\|G\|_B = \max\{\|Gx\|_B \mid \|x\|_B = 1.\} \quad (34)$$

The Frobenius norm of a matrix G (not necessarily square) is

$$\|G\|_F = \left(\sum_i \|g_i\|_2^2\right)^{1/2} \quad (35)$$

where g_i is the i th column vector of G being square and of the same order as B and $G = (g_i)$. It is known (Lancaster and Tismenetsky, 1985) that if G^+ and G^* are the Moore-Penrose inverse and the conjugate of any matrix G , then we have

$$\begin{aligned} GG^+G &= G \\ G^+GG^+ &= G^+ \\ (GG^+)^* &= GG^+ \\ (G^+G)^* &= G^+G. \end{aligned} \quad (36)$$

It is also known that for a square matrix G and a vector x we have

$$x^t \cdot Gx \geq \lambda_{\min} \|x\|_2^2 \quad (37)$$

$$x^t \cdot Gx \leq \lambda_{\max} \|x\|_2^2 \quad (38)$$

where λ_{min} and λ_{max} are respectively the minimum and maximum eigenvalues of the matrix G .

We are given two nonzero vectors x and y and a matrix G where $G \in R^{n \times m}$ is one-to-one and onto, i.e., G is invertible. Now we consider the following transformation:

$$x = Gy \quad (39)$$

The results of the following three propositions about the linear transformation (39) are used in the proof of convergence of the algorithms.

Proposition A.1

$$1) \|x\|_2 \leq \|G\|_F \cdot \|y\|_2 \quad (40)$$

$$2) \|x\|_2 \geq \|G^+\|_F^{-1} \cdot \|y\|_2 \quad (41)$$

Proof.

1) By definition of the norm consistency, we have

$$\|x\|_2 = \|Gy\|_2 \leq \|G\|_F \cdot \|y\|_2$$

2) Since G^+ is the Moore-Penrose inverse of G , we have $y = G^+x$ and

$$\|y\|_2 = \|G^+x\|_2 \leq \|G^+\|_F \cdot \|x\|_2$$

$$\|x\|_2 \geq \|G^+\|_F^{-1} \cdot \|y\|_2. \quad \square$$

Proposition A.2

$$1) \|x\|_2 \leq \lambda_{min}^{-1/2} \|x\|_B \quad (42)$$

$$2) \|x\|_2 \geq \lambda_{max}^{-1/2} \|x\|_B \quad (43)$$

where λ_{min} and λ_{max} are the minimum and maximum eigenvalues of the symmetric positive definite matrix B .

Proof.

By definition, we have

$$\|x\|_2 = (\sum_i \alpha_i^2)^{1/2} \text{ and } \|x\|_B = (x \cdot Bx)^{1/2} = (\sum_i \alpha_i^2 \lambda_i)^{1/2}$$

where α_i is the i th principal component of x and λ_i is the i th eigenvalue of B . Since $\lambda_{\min} > 0$, we have

$$\|x\|_B \geq \lambda_{\min}^{1/2} (\sum_i \alpha_i^2)^{1/2} = \lambda_{\min}^{1/2} \|x\|_2 \leftrightarrow \|x\|_2 \leq \lambda_{\min}^{-1/2} \|x\|_B$$

$$\|x\|_B \leq \lambda_{\max}^{1/2} (\sum_i \alpha_i^2)^{1/2} = \lambda_{\max}^{1/2} \|x\|_2 \leftrightarrow \|x\|_2 \geq \lambda_{\max}^{-1/2} \|x\|_B. \quad \square$$

Proposition A.3

$$1) \|x\|_2^2 \leq \beta \|y\|_B^2 \quad (44)$$

$$2) \|x\|_2^2 \geq \gamma \|y\|_B^2 \quad (45)$$

where B is a symmetric and positive definite matrix, $\beta = \|G\|_F^2 \lambda_{\min}^{-1}$ and $\gamma = \|G^+\|_F^{-2} \lambda_{\max}^{-1}$.

Proof.

$$1) \|x\|_2^2 \leq \|G\|_F^2 \cdot \|y\|_2^2 \leq \|G\|_F^2 \lambda_{\min}^{-1} \|y\|_B^2 = \beta \|y\|_B^2.$$

$$2) \|x\|_2^2 \geq \|G^+\|_F^{-2} \cdot \|y\|_2^2 \geq \|G^+\|_F^{-2} \lambda_{\max}^{-1} \|y\|_B^2 = \gamma \|y\|_B^2. \quad \square$$

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